ISRAEL JOURNAL OF MATHEMATICS 123 (2001), 61–92

SOBOLEV INEQUALITIES OF EXPONENTIAL TYPE

BY

D. E. EDMUNDS

Centre for Mathematical Analysis and Its Applications
University of Sussex, Falmer, Brighton BN1 9QH, U.K.
e-mail: D.E.Edmunds@sussex.ac.uk

AND

R. HURRI-SYRJÄNEN*

Department of Mathematics, University of Helsinki
P.O. Box 4, FIN-00014, Finland
e-mail: ritva.hurri-syrjanen@helsinki.fi

ABSTRACT

We give sufficient conditions for domains to satisfy Sobolev inequalities of single exponential type. Earlier work in this area imposed more stringent conditions on the domains and is thus contained in our results. Moreover, the class of functions considered is based on $L^n \log^{an} L$ with $a < 1-1/n$, n being the dimension of the underlying space. The limiting case $a = 1-1/n$ gives rise to an inequality of double exponential type which is shown to be valid in a large class of irregular domains. This inequality is new even in smooth domains.

1. Introduction

In the last decade, considerable attention has been paid to the validity of classical Sobolev inequalities and embeddings when the underlying bounded domain D in R^n ($n \geq 2$) need not have a smooth boundary ∂D: see, for example, [8], [6], [12]. Most of this work has been concerned with the situation in which the target

* The second author was partially supported by a grant from Magnus Ehrnrooth Foundation.
Received January 27, 2000

space of the embedding is of Lebesgue or Hölder type. Much less is known about the position in limiting cases. Thus if ∂D is smooth, it is a familiar fact that the Sobolev space $W^1_n(D)$ is embedded in the Orlicz space $L_\phi(D)$ with Young function ϕ with values which behave like $\exp(t^{n/(n-1)})$ for large values of the argument t: see, for example, [23], [19], [22], [1]. This embedding for Hölder domains has been considered in [21]. The case when the Sobolev space $W^1_{n+\epsilon}(D)$ is embedded in the Orlicz space $L_\phi(D)$ with $\phi(t) = \exp(t^{(n+\epsilon)'})$, $\epsilon > 0$, is studied in [4]. For any exponent p, $1 < p < \infty$, we write $p' = p/(p-1)$. We consider here, on the one hand, spaces of functions that are larger than $W^1_n(D)$, but are contained in $\bigcap_{1<p<n} W^1_p(D)$, and, on the other hand, also spaces of functions that are smaller than $W^1_n(D)$, but each $W^1_p(D)$ with $p > n$ is contained in them.

Our principal object in this paper is to give conditions on D which are sufficient to ensure Sobolev inequalities yet allow the boundary of D to be quite rough. The first main result concerns a c_0-John domain, by which we mean a domain D in R^n such that there exist $x_0 \in D$ and $c_0 \in (0,1]$ with the property that every $x \in D$ can be joined to x_0 by a rectifiable curve $\gamma : [0,l] \to D$, parametrized by arc length, with $d(\gamma(t), \partial D) \geq c_0 t$ for all $t \in [0,l]$. The class of such domains is wide, and includes Lipschitz domains and the Koch snowflake. We show that if D is a bounded c_o-John domain and u is a function on D such that

$$I(a,D) := \left(\int_D |\nabla u(x)|^n \log^{an}(e + |\nabla u(x)|)\,dx\right)^{1/n} < \infty$$

for some $a < 1 - 1/n$, then there are constants $c_i = c_i(a, c_0, |D|, n)$ $(i = 1,2)$ such that

$$\int_D \exp\left(\frac{|u(x) - u_D|}{c_1 I(a,D)}\right)^\alpha dx \leq c_2,$$

where $\alpha = n/(n-1-an)$ and $u_D = |D|^{-1}\int_D u(x)\,dx$. When ∂D is smooth, the special case $a = 0$ corresponds to the Trudinger embedding and $a < 0$ to the result obtained by Fusco, Lions and Sbordone [9]. We generalize these results to the cases $0 < a < 1 - 1/n$, too. See [7] and [5] for related work for arbitrary values of a and for additional references.

We also show that an inequality of this type holds for sets which may have even more irregular boundaries, such as those which satisfy an appropriate type of quasi-hyperbolic boundary condition [10]. We give geometric criteria for domains sufficient for Sobolev inequalities of single exponential type to be satisfied. In particular, our results extend the class of domains where the classical Trudinger inequality is known to hold.

In a limiting case, corresponding to $a = 1 - 1/n$, we show that a double exponential inequality holds in smooth and certain nonsmooth domains $D \subset R^n$: given any positive constants A_1 and $A_2 < 1$ such that $A_1 A_2 < 2e^{-1}$, there is $A_3 > 0$, which does not depend on $|\nabla u|$, such that

$$\int_D \exp\left[A_1 \exp\left(A_2\left(\frac{|u(x) - u_D|}{I(D)}\right)^{n/(n-1)}\right)\right] dx \le A_3,$$

where

$$I(D) := \left(\int_D |\nabla u(x)|^n \log^{n-1}(e + |\nabla u(x)|)\, dx\right)^{1/n} < \infty.$$

For embeddings into spaces of double exponential type see also [7]. This inequality is new even in balls as far as we know.

The notation used and some preliminary results are given in Section 2. The next section deals with John domains and finally Sections 4 and 5 contain material on more irregular domains.

2. Notation and preliminary results

Throughout this paper, D will stand for a bounded domain in R^n, $n \ge 2$, unless otherwise stated. Given any sets $A, B \subset R^n$ and any $x \in R^n$, $d(x, \partial A)$ denotes the distance from x to the boundary ∂A. If A is measurable, its n-dimensional Lebesgue measure will be denoted by $|A|$.

The notation $a \preceq b$ is an abbreviation for the inequality $a \le cb$, where c is a constant independent of a and b. By $a \simeq b$ is meant the inequality $c_1 a \le b \le c_2 a$, where c_1, c_2 are constants independent of a and b. We let $c(*, \dots, *)$ denote a constant which depends only on the quantities appearing in the parentheses.

The notation considering Sobolev spaces is standard.

Define $⨍_D u = u_D = |D|^{-1} \int_D u(x)\, dx$.

A domain D is called an (α, β)-John domain, $0 < \alpha \le \beta < \infty$, if there exists x_0 such that each $x \in D$ can be joined to x_0 by a rectifiable curve γ: $[0, l] \to D$, parametrized by arc length, with $l \le \beta$ and

$$d(\gamma(t), \partial D) \ge \frac{\alpha}{\beta} t, \quad t \in [0, l].$$

By a c_0-John domain D we mean that D is an (α, β)-John domain with some constants α and β such that $\alpha/\beta = c_0$.

Convex domains and Lipschitz domains are John domains, and so is a Koch snowflake domain, [16].

Other definitions needed are introduced when we first use them.

LEMMA 2.1: *Let $D \subset R^n$, $n \geq 2$, be a bounded c_0-John domain. Let $p, q \in [1, \infty]$ be such that*

$$0 \leq 1/p - 1/q < 1/n.$$

Then

$$\left(\int_D |u(x) - u_D|^q \, dx\right)^{\frac{1}{q}}$$
$$\leq c(n)\omega_n^{1-1/n}\left(\frac{1 - 1/p + 1/q}{1/n - 1/p + 1/q}\right)^{1-\frac{1}{p}+\frac{1}{q}} c_0^{-1}|D|^{1/n-1/p+1/q}\left(\int_D |\nabla u(x)|^p \, dx\right)^{\frac{1}{p}}$$

for all $u \in W_p^1(D)$, where ω_n is the $(n-1)$-dimensional Lebesgue measure of the unit sphere S^{n-1} in R^n.

Proof: The result is contained in [3, Theorem 5.1]. It also follows from [15, Lemma 3.3.] and [11, Lemma 7.12]. ∎

We use Lemma 2.1 by choosing $q = (n-p)^{-1}$.

Remark 2.2: Let $0 < \epsilon \leq 1/n$. Let us choose $q = \frac{1}{\epsilon}$ and $p = n - \epsilon$ in Lemma 2.1. Then

$$1/p - 1/q < 1/n$$

and there is a constant $c(n)$ such that

$$\left(\frac{1 - 1/p + 1/q}{1/n - 1/p + 1/q}\right)^{1-1/p+1/q} \leq c(n)\epsilon^{1/n-1}.$$

Thus, by Lemma 2.1

$$\left(\int_D |u(x) - u_D|^{1/\epsilon} \, dx\right)^{\epsilon}$$
$$\leq c(n)\omega_n^{1-1/n}c_0^{-1}|D|^{1/n-1/(n-\epsilon)+\epsilon}\epsilon^{1/n-1}\left(\int_D |\nabla u(x)|^{n-\epsilon} \, dx\right)^{1/(n-\epsilon)}$$

and so

$$\left(\fint_D |u(x) - u_D|^{1/\epsilon} \, dx\right)^{\epsilon}$$
$$\leq c(n)\omega_n^{1-1/n}c_0^{-1}|D|^{1/n}\epsilon^{1/n-1}\left(\fint_D |\nabla u(x)|^{n-\epsilon} \, dx\right)^{1/(n-\epsilon)}.$$

The idea in the following lemma is adapted from [9].

LEMMA 2.3: *Let $D \subset R^n$, $n \geq 2$, be a bounded c_0-John domain. Let $\sigma \geq 0$, $\alpha = n/(n-1+\sigma)$ and $0 < \epsilon \leq 1/n$. Then*

$$\left(\fint_D |u(x) - u_D|^{1/\epsilon}\, dx\right)^{\epsilon} \epsilon^{1/\alpha} \leq c(n) c_0^{-1} |D|^{1/n} \sup_{0<\epsilon\leq 1} \left(\epsilon^{\sigma} \fint_D |\nabla u(x)|^{n-\epsilon}\, dx\right)^{1/(n-\epsilon)},$$

whenever $u \in C^1(D)$.

Proof: By Remark 2.2

$$\begin{aligned}
&\left(\fint_D |u(x) - u_D|^{1/\epsilon}\, dx\right)^{\epsilon} \epsilon^{1/\alpha} \\
&\leq c(n) c_0^{-1} |D|^{1/n} \epsilon^{1/\alpha - 1 + 1/n - \sigma/n} \left(\epsilon^{\sigma} \fint_D |\nabla u(x)|^{n-\epsilon}\, dx\right)^{1/(n-\epsilon)} \epsilon^{-\sigma\epsilon/n(n-\epsilon)} \\
&= c(n) c_0^{-1} |D|^{1/n} \left(\epsilon^{\sigma} \fint_D |\nabla u(x)|^{n-\epsilon}\, dx\right)^{1/(n-\epsilon)} \epsilon^{-\sigma\epsilon/n(n-\epsilon)} \\
&\leq c(n) c_0^{-1} |D|^{1/n} \sup_{0<\epsilon\leq 1} \left(\epsilon^{\sigma} \fint_D |\nabla u(x)|^{n-\epsilon}\, dx\right)^{1/(n-\epsilon)},
\end{aligned}$$

since $\lim_{\epsilon\to 0+} \epsilon^{-\epsilon\sigma/(n^2-1)} = 1$ and $\sigma \geq 0$.

The claim follows. ∎

We agree on some notation in Remarks 2.4.

Remarks 2.4: 1. Assume that $n \in N$, $n \geq 2$. Let $\sigma > -n+1$ and $\alpha = n/(n-1+\sigma)$. The Orlicz function $t \mapsto \exp t^{\alpha} - 1$ is denoted by ψ_0. We write

$$\psi_{0_1}(t) = \exp t^{\alpha} - \sum_{j=0}^{j_0-1} \frac{t^{\alpha j}}{j!},$$

where j_0 is an arbitrary positive integer. Hence $\psi_0(t) \simeq \psi_{0_1}(t)$ for all $t \geq 1$. By [2, Theorem 8.12 b] it is enough to consider ψ_{0_1} when estimating Orlicz norms.

2. We agree on some notation from now on. Once we have fixed a suitable domain D in R^n and an admissible function u there, we denote

$$\int_A |x-y|^{1-n} |\nabla u(y)|\, dy =: R(|\nabla u|, A)(x),$$

where $x \in D$, and

$$\left(\int_A |\nabla u(y)|^n \log(e + |\nabla u(y)|)^{an}\, dy\right)^{1/n} =: I(a, A)$$

where $A \subset D$ is a measurable set and $a \in (-\infty, 1-1/n]$ is a parameter. If $a = 1-1/n$, we write briefly $I(1-1/n, A) =: I(A)$.

3. If a domain D has the property that for some $p \in [1,\infty]$ and $q \in [1,\infty]$ there is a constant C such that for all u in the Sobolev space $W^1_p(D)$ the inequality

$$\Big(\int_D |u(x)-u_D|^q\,dx\Big)^{1/q} \le C\Big(\int_D |\nabla u(x)|^p\,dx\Big)^{1/p}$$

holds, then we say that D is a (q,p)-Poincaré domain.

When D is a (p,p)-Poincaré domain with some $p \in [1,n]$ and there is a function u such that $I(a,D) < \infty$, then it is enough to consider a function $u \in C^1(D) \cap W^1_1(D)$. Namely, if $a \le 0$ and $I(a,D) < \infty$, then $u \in L^1_p = \{u : |\nabla u| \in L^p(D)\}$ for all $p < n$. Since D is a (p,p)-Poincaré domain for some $p \in [1,n]$, we have $L^1_p(D) = W^1_p(D)$, and thus $u \in W^1_1(D)$. If $a > 0$ and $I(a,D) < \infty$, then $u \in L^1_n(D)$ and also $u \in W^1_n(D)$. For D is an (n,n)-Poincaré domain, since D is a (p,p)-Poincaré domain for some $p \in [1,n]$, Thus $u \in W^1_1(D)$ also in the case $a > 0$.

The above yields that $u_D < \infty$.

We recall that a John domain is a (p,p)-Poincaré domain for all p, $1 \le p < \infty$.

We need the following lemmata when studying the cases $a \in (0, 1-1/n]$. The idea of Lemma 2.5 is from [18, Theorem 1].

LEMMA 2.5: *Let D be a bounded domain in R^n and let $u \in W^1_1(D)$ satisfy $I(a,D) < \infty$ with some $a \in (0,1-1/n]$. If $2 < a_1 < a_2$ and $q > n$, then there exists a constant $c = c(n) < \infty$ such that*

$$\begin{aligned}\Big(\int_D |R(|\nabla u|,D)(x)|^q\,dx\Big)^{1/q} &\le c(n)|D|^{1/q}\Big(|D|^{1/n}a_1\\ &+\Big(\int_1^{a_2}\frac{dr}{r\log(e+r)^{an/(n-1)}}\Big)^{(n-1)/n} I(a,\{y\in D : a_1 < |\nabla u(y)| < a_2\})\\ &+\frac{q^{(n-1)/n}}{(\log(e+a_2))^a} I(a,\{y\in D : a_2 \le |\nabla u(y)|\})\Big).\end{aligned}$$

Proof: We may assume that $u \in C^1(D)$. The standard potential estimate, [11, Section 7], yields

$$\begin{aligned}(2.1)\qquad &\Big(\int_D\Big(\int_{\{y\in D:|\nabla u(y)|\le a_1\}} |x-y|^{1-n}|\nabla u(y)|\,dy\Big)^q dx\Big)^{1/q}\\ &\le \Big(\int_D\Big(\int_D |x-y|^{1-n}a_1\,dy\Big)^q dx\Big)^{1/q}\\ &\le c(n)a_1|D|^{1/n+1/q}.\end{aligned}$$

For each $j \in N$ we set

$$E_j = \{y \in D : |\nabla u(y)| > a_1; 2^{-j}a_2 \le |\nabla u(y)| < 2^{-j+1}a_2\}.$$

Then,

$$R(|\nabla u|, E_j) \le 2^{-j+1}a_2 \int_{E_j} |x-y|^{1-n}\,dy \le c(n)a_2 2^{-j}|E_j|^{1/n}.$$

Hence, by Hölder's inequality with $(n, \frac{n}{n-1})$,

$$\begin{aligned}
&R(|\nabla u|, \{y \in D : a_1 < |\nabla u(y)| < a_2\}) = \sum_{j=1}^{\infty} R(|\nabla u|, E_j)\\
&\le c(n)\sum_{j=1}^{\infty} 2^{-j}a_2|E_j|^{1/n}\\
&= c(n)\sum_{j=1}^{\infty} 2^{-j}a_2|E_j|^{1/n}(\log(e+2^{-j}a_2))^{a}(\log(e+2^{-j}a_2))^{-a}\\
&\le c(n)\left(\sum_{j=1}^{\infty} 2^{-jn}a_2^n|E_j|(\log(e+2^{-j}a_2))^{an}\right)^{1/n}\\
&\quad \times \left(\sum_{j=1}^{\infty}(\log(e+2^{-j}a_2))^{-an/(n-1)}\right)^{(n-1)/n}.
\end{aligned}$$

Here,

$$\sum_{j=1}^{\infty} 2^{-jn}a_2^n|E_j|(\log(e+2^{-j}a_2))^{an} \le I(a, \{y \in D : a_1 < |\nabla u(y)| < a_2\})^n,$$

and

$$\begin{aligned}
\sum_{j=1}^{\infty}(\log(e+2^{-j}a_2))^{-an/(n-1)} &\le 2\sum_{j=1}^{\infty}\int_{a_2/2^{j+1}}^{a_2/2^j} \frac{dr}{r\log(e+r)^{an/(n-1)}}\\
&\le 2\int_1^{a_2} \frac{dr}{r\log(e+r)^{an/(n-1)}}.
\end{aligned}$$

By combining the above we obtain

$$\begin{aligned}
(2.2)\qquad &\left(\int_D |R(|\nabla u|, \{y \in D : a_1 < |\nabla u(y)| < a_2\})(x)|^q\,dx\right)^{1/q}\\
&\le c(n)|D|^{1/q}\left(\int_1^{a_2} \frac{dr}{r\log(e+r)^{an/(n-1)}}\right)^{(n-1)/n}\\
&\quad \times I(a, \{y \in D : a_1 < |\nabla u(y)| < a_2\}).
\end{aligned}$$

The standard potential estimates imply

$$
\begin{aligned}
(2.3)\qquad &\left(\int_D |R(|\nabla u|, \{y \in D : |\nabla u(y)| \geq a_2\})(x)|^q \, dx\right)^{1/q} \\
&\leq c(n)|D|^{1/q} q^{(n-1)/n} \left(\int_{\{y \in D : |\nabla u(y)| \geq a_2\}} |\nabla u(y)|^n \, dy\right)^{1/n} \\
&\leq c(n)|D|^{1/q} \frac{q^{(n-1)/n}}{(\log(e+a_2))^a} I(a, \{y \in D \, : a_2 \leq |\nabla u(y)|\}).
\end{aligned}
$$

Estimates (2.1)–(2.3) give the claim. ∎

Remark 2.6: For $a_2 > 1$ and $a \in (0, 1-1/n]$

$$
\int_1^{a_2} \frac{dr}{r \log(e+r)^{an/(n-1)}} \geq \int_1^{a_2} \frac{dr}{r \log(e+a_2)^{an/(n-1)}} = \frac{\log a_2}{\log(e+a_2)^{an/(n-1)}}.
$$

LEMMA 2.7: *Let $q > n$. Then*

$$
\left(\int_1^{e^q} \frac{dr}{r \log(e+r)^{an/(n-1)}}\right)^{(n-1)/n} \leq (10 \log(e+q))^{(n-1)/n},
$$

if $a = 1 - 1/n$, and

$$
\left(\int_1^{e^q} \frac{dr}{r \log(e+r)^{an/(n-1)}}\right)^{(n-1)/n} \leq \left(\frac{q^{1-an/(n-1)}}{1 - \frac{an}{n-1}}\right)^{(n-1)/n},
$$

if $0 \leq a < 1 - 1/n$.

Proof: A simple calculation. ∎

LEMMA 2.8: *For any $A > e$.*

$$
\sum_{j=0}^{\infty} \frac{1}{j!} A^j (\log(e+j))^j \leq A^{2A}.
$$

Proof: [18, Lemma 7]. ∎

3. John domains

The main results in this section are Theorems 3.2, 3.3 and 3.4. First we formulate and prove the following exponential-type inequality.

THEOREM 3.1: *Let $D \subset R^n$ be a bounded c_0-John domain. Let $u \in C^1(D)$ satisfy, for some $\sigma \geq 0$,*

$$M = \sup_{0<\epsilon\leq 1}\left(\epsilon^\sigma \fint_D |\nabla u(x)|^{n-\epsilon}\,dx\right)^{1/(n-\epsilon)} < \infty.$$

Then, if $\alpha = n/(n-1+\sigma)$, there exist constants $c_i = c_i(n,\sigma,c_0)$, $i=1,2$, such that

$$\fint_D \exp\left(\frac{|u(x)-u_D|}{c_1 M|D|^{1/n}}\right)^\alpha dx \leq c_2.$$

Proof: Lemma 2.3 with $q = \alpha j = 1/\epsilon$ implies

$$\begin{aligned}\fint_D \exp\left(\frac{|u(x)-u_D|\tau}{M|D|^{1/n}}\right)^\alpha dx &\simeq \sum_{j=n}^\infty \frac{\tau^{\alpha j}}{j!}\frac{\fint_D |u(x)-u_D|^{\alpha j}\,dx}{M^{\alpha j}|D|^{\alpha j/n}}\\ &\leq \sum_{j=n}^\infty \frac{\tau^{\alpha j}}{j!}(c(n)c_0^{-1})^{\alpha j}(\alpha j)^j \leq c < \infty,\end{aligned}$$

whenever τ is chosen such that

$$(\tau c(n)c_0^{-1})^\alpha \alpha < e^{-1}. \qquad ∎$$

THEOREM 3.2: *Let D be a bounded c_0-John domain in R^n and let u satisfy*

$$I(a,D) = \left(\int_D |\nabla u(x)|^n \log(e+|\nabla u(x)|)^{an}\,dx\right)^{1/n} < \infty$$

for some $a \leq 0$. Then there exist constants $c_i = c_i(a, c_0, |D|, n)$, $i=1,2$, such that

$$\fint_D \exp\left(\frac{|u(x)-u_D|}{c_1 I(a,D)}\right)^\alpha dx \leq c_2,$$

where $\alpha = n/(n-1-an)$.

Proof: We consider first the case $a<0$ and write $\sigma = -an$. By Remark 2.4(3) it is enough to consider functions $u \in C^1(D)\cap W_1^1(D)$. We may assume that

$$I(a,D) \leq 1.$$

As in the proof for Theorem 3.1

$$\fint_D \exp\left(\frac{\tau|u(x)-u_D|}{I(a,D)}\right)^\alpha dx \simeq \sum_{j=n}^\infty \frac{\tau^{\alpha j}}{j!}\frac{\fint_D |u(x)-u_D|^{\alpha j}\,dx}{I(a,D)^{\alpha j}} \tag{3.1}$$

with a number τ which will be chosen later. Lemma 2.3 with $q=\epsilon^{-1}$ yields

$$\begin{aligned}(3.2)\qquad & ⨍_D |u(x)-u_D|^q\,dx\epsilon^{1/\alpha\epsilon} \\ &\le (c(n)c_0^{-1}|D|^{1/n})^q \sup_{0<\epsilon\le 1}(\epsilon^\sigma ⨍_D |\nabla u(x)|^{n-\epsilon}\,dx)^{1/\epsilon(n-\epsilon)} \\ &\le \left(c(n)c_0^{-1}|D|^{1/n}\sup_{0<\epsilon\le 1}(\epsilon^\sigma ⨍_D |\nabla u(x)|^{n-\epsilon}\,dx)^{1/(n-\epsilon)}\right)^q.\end{aligned}$$

We now consider

$$M=\sup_{0<\epsilon\le 1}\left(\epsilon^\sigma ⨍_D |\nabla u(x)|^{n-\epsilon}\,dx\right)^{1/(n-\epsilon)}$$

and study the sets $\{x : |\nabla u(x)|\le e\}$ and $\{x : |\nabla u(x)|\ge e\}$ separately.

If $|\nabla u(x)|\ge e$, then

$$\log(e+|\nabla u(x)|)\le\frac{(2|\nabla u(x)|)^{\epsilon/\sigma}}{\epsilon/\sigma},$$

and thus

$$\begin{aligned}(3.3)\qquad & \epsilon^\sigma\frac{1}{|D|}\int_{D\cap\{x:|\nabla u(x)|\ge e\}}\frac{|\nabla u(x)|^n}{|\nabla u(x)|^\epsilon}\,dx=|D|^{-1}\sigma^\sigma 2^\epsilon \\ &\times\int_{D\cap\{x:|\nabla u(x)|\ge e\}}\frac{|\nabla u(x)|^n}{\log(e+|\nabla u(x)|))^\sigma}\left(\frac{\log(e+|\nabla u(x)|)}{\frac{(2|\nabla u(x)|)^{\epsilon/\sigma}}{\epsilon/\sigma}}\right)^\sigma dx \\ &\le\frac{\sigma^\sigma 2^\epsilon}{|D|}I(a,D\cap\{x : |\nabla u(x)|\ge e\})^n \\ &\le\frac{\sigma^\sigma 2^{1/n}}{|D|}I(a,D)^n.\end{aligned}$$

Since

$$I(a,D)\le 1\quad\text{and}\quad\frac{1}{n^2}\le\frac{1}{n(n-\epsilon)}\le\frac{1}{n^2-1},$$

and we may assume that $|D|\le 1$, estimate (3.3) yields

$$\begin{aligned}(3.4)\qquad & \left(\epsilon^\sigma\frac{1}{|D|}\int_{D\cap\{x:|\nabla u(x)|\ge e\}}\frac{|\nabla u(x)|^n}{|\nabla u(x)|^\epsilon}\,dx\right)^{1/(n-\epsilon)} \\ &\le\left(\sigma^\sigma 2^{1/n}\frac{1}{|D|}\int_{D\cap\{x:|\nabla u(x)|\ge e\}}\frac{|\nabla u(x)|^n}{|\nabla u(x)|^\epsilon}\,dx\right)^{1/(n-\epsilon)} \\ &\le\frac{\sigma^{\sigma c(n)}2^{1/(n^2-1)}}{|D|^{n/(n^2-1)}}I(a,D).\end{aligned}$$

Thus, from (3.2),

$$(3.5)\qquad \fint_D |u(x)-u_D|^q\,dx\epsilon^{1/\alpha\epsilon} \le \left(c(n)c_0^{-1}|D|^{-1/(n^2-1)}\sigma^{\sigma c(n)}I(a,D)\right)^q.$$

If $|\nabla u(x)| \le e$, then Hölder's inequality with $(n/(n-\epsilon), n/\epsilon)$ and the inequalities

$$\log(e+|\nabla u(x)|) \le \log(e+e) = \log 2e$$

imply

$$\begin{aligned}(3.6)\qquad &\epsilon^\sigma |D|^{-1}\int_{D\cap\{x:|\nabla u(x)|\le e\}} |\nabla u(x)|^{n-\epsilon}\,dx\\ &\le |D|^{-1}\int_{D\cap\{x:|\nabla u(x)|\le e\}} |\nabla u(x)|^{n-\epsilon}\,dx\\ &\le \left(\int_D |\nabla u(x)|^n\,dx\right)^{(n-\epsilon)/n}|D|^{-1+\epsilon/n}\\ &\le \left(\int_D |\nabla u(x)|^n \frac{\log^{-\sigma}(e+|\nabla u(x)|)}{\log^{-\sigma}(2e)}dx\right)^{(n-\epsilon)/n}|D|^{-1+\epsilon/n},\end{aligned}$$

since $\sigma > 0$. Thus

$$(3.7)\qquad \fint_D |u(x)-u_D|^q\,dx\epsilon^{1/\alpha\epsilon} \le \left(\frac{c(n)c_0^{-1}|D|^{1/n}|D|^{-1/n}}{\log^{-\sigma/n}(2e)}I(a,D)\right)^q.$$

Recall $\epsilon = 1/q = 1/\alpha j$. Hence (3.6) and (3.7) imply

$$\begin{aligned}(3.8)\ &\fint_D \exp\left(\frac{\tau|u(x)-u_D|}{I(a,D)}\right)^\alpha dx\\ &\simeq \sum_{j=n}^\infty \frac{\tau^{\alpha j}}{j!}(\alpha j)^j\left(c(n)c_0^{-1}\left(\sigma^{\sigma c(n)}|D|^{-1/(n^2-1)}+\log^{\sigma/n}(2e)\right)\right)^q \le c < \infty\end{aligned}$$

for some τ. Theorem 3.2 is proved whenever $a < 0$.

The case $a = 0$ follows from (3.1) and (3.2). We just use Hölder's inequality with $(\frac{n}{n-\epsilon}, \frac{n}{\epsilon})$ in (3.2). ∎

LEMMA 3.1: *Let $D \subset R^n$ be a bounded c_0-John domain and let $q > n$ and $2 < a_1$. Then*

$$\begin{aligned}&\left(\int_D |u(x)-u_D|^q\,dx\right)^{1/q}\\ &\le c_0^{-1}c(n)|D|^{1/q}\left(|D|^{1/n}a_1 + c(a,n)I(\{y\in D: a_1 \le |\nabla u(y)|\})q^{(n-1-an)/n}\right),\end{aligned}$$

whenever u satisfies $I(a,D)<\infty$ with $a\in[0,1-1/n)$.

Proof: Since D is a John domain, by [15, Lemma 3.3]

$$\left(\int_D |u(x)-u_D|^q\,dx\right)^{1/q} \le c_0c(n)\left(\int_D\left(\int_D |x-y|^{1-n}|\nabla u(y)|\,dy\right)^q dx\right)^{1/q}.$$

Lemma 2.5 and Remark 2.6 imply for $a_2=e^q>a_1$

$$\begin{aligned}
&\left(\int_D\left(\int_D |x-y|^{1-n}|\nabla u(y)|\,dy\right)^q dx\right)^{1/q}\\
&\le c(n)|D|^{1/q}\Bigg(|D|^{1/n}a_1+\left(\int_1^{a_2}\frac{dr}{r\log(e+r)^{an/(n-1)}}\right)^{(n-1)/n}\\
&\qquad\qquad\times I(a,\{y\in D: a_1<|\nabla u(y)|<a_2\})\\
&\qquad+\frac{q^{(n-1)/n}}{(\log(e+a_2))^a}I(a,\{y\in D: a_2\le|\nabla u(y)|\})\Bigg)\\
&\le c(n)|D|^{1/q}\Bigg(|D|^{1/n}a_1+\left(\int_1^{a_2}\frac{dr}{r\log(e+r)^{an/(n-1)}}\right)^{(n-1)/n}\\
&\qquad+2I(a,\{y\in D: a_1\le|\nabla u(y)|\})\Bigg).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left(\int_D\left(\int_D |x-y|^{1-n}|\nabla u(y)|\,dy\right)^q dx\right)^{1/q}\\
&\le c(n)|D|^{1/q}\Big(|D|^{1/n}a_1+c(a,n)I(\{y\in D: a_1\le|\nabla u(y)|\})q^{(n-1-an)/n}\Big).
\end{aligned}$$

The claim follows. ∎

THEOREM 3.3: *Let D be a bounded c_0-John domain in R^n and let $I(a,D)<\infty$ with $a\in[0,1-1/n)$. Then*

$$\int_D \exp\left[A\left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{n/(n-1-an)}\right]dx<\infty$$

for any non-negative constant $A<e^{-1}$.

Proof: By Remark 2.4(3) we may assume that $u\in C^1(D)\cap W_1^1(D)$. We use the modified power series expansion of the exponential function (Remarks 2.4)

$$\int_D \exp A\left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{n/(n-1-an)} dx \simeq \sum_{j=j_0}^{\infty} \frac{A^j}{j!} \int_D \left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{nj/(n-1-an)} dx$$

with a constant $A > 0$. By Lemma 3.1 and with $q = nj/(n-1-an)$ we obtain

$$\begin{aligned}\left\{\int_D \left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{nj/(n-1-an)} dx\right\}^{(n-1-an)/nj} &\le (c_0^{-1}c(n))\frac{|D|^{1/q}}{I(a,D)}\Bigg(|D|^{1/n}a_1 + \left(\frac{q^{1-an/(n-1)}}{1-\frac{an}{n-1}}\right)^{(n-1)/n} \\ &\qquad \times I(\{y \in D : a_1 \le |\nabla u(y)|\})\Bigg) \\ &= c_0^{-1}c(a,n)|D|^{1/q}j^{(n-1-an)/n}\Bigg(\frac{a_1|D|^{1/n}}{I(a,D)c_1(a,n)j^{(n-1-an)/n}} \\ &\qquad + \frac{I(\{y \in D : a_1 \le |\nabla u(y)|\})}{I(a,D)}\Bigg).\end{aligned}$$

We choose a_1 so large that

$$\frac{I(\{y \in D : a_1 \le |\nabla u(y)|\})}{I(a,D)} \le (2c_0^{-1}c(a,n))^{-1}.$$

Then for this a_1 we fix j_0 so large that

$$\frac{a_1|D|^{1/n}c_0^{-1}c(a,n)}{I(a,D)c_1(a,n)j^{(n-1-an)/n}} \le \frac{1}{2}$$

with all $j \ge j_0$. Thus

$$\sum_{j=j_0}^{\infty} \int_D \frac{A^j}{j!}\left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{nj/(n-1-an)} dx \le |D| \sum_{j=j_0}^{\infty} \frac{A^j}{j!} j^j < \infty,$$

whenever $0 < A < e^{-1}$. Then for any non-negative constant $A < e^{-1}$,

$$\int_D \exp\left[A\left(\frac{|u(x)-u_D|}{I(a,D)}\right)^{n/(n-1-an)}\right] dx < \infty. \qquad \blacksquare$$

LEMMA 3.2: *Let $D \subset R^n$ be a bounded c_0-John domain and let $q > n$ and $2 < a_1$. Then*

$$\left(\int_D |u(x) - u_D|^q \, dx\right)^{1/q} \leq c_0^{-1} c(n) |D|^{1/q} \Bigg(|D|^{1/n} a_1 + \Big(\log(e+q)\Big)^{(n-1)/n} \times I(\{y \in D : a_1 \leq |\nabla u(y)|\}) \Bigg),$$

whenever $u \in W_1^1(D)$ satisfies $I(1-1/n, D) < \infty$.

Proof: Since D is a John domain, from [15, Lemma 3.3] we obtain

$$\left(\int_D |u(x) - u_D|^q \, dx\right)^{1/q} \leq c_0^{-1} c(n) \left(\int_D \left(\int_D |x-y|^{1-n} |\nabla u(y)| \, dy\right)^q dx\right)^{1/q}.$$

The claim follows from Lemma 2.5, Remark 2.6 with $a_2 = e^q$ and Lemma 2.7. ∎

THEOREM 3.4: *Let D be a bounded c_0-John domain in R^n and let $u \in W_1^1(D)$ satisfy $I(1-1/n, D) = I(D) < \infty$. Given any positive constants A_1 and $A_2 < 1$ such that $A_1 A_2 < 2e^{-1}$, then there is $A_3 > 0$ such that*

$$\int_D \exp\left[A_1 \exp\left(A_2 \left(\frac{|u(x) - u_D|}{I(D)}\right)^{n/(n-1)}\right)\right] dx \leq A_3 < \infty.$$

The constant A_3 does not depend on the gradient of u.

Proof: Let $u \in W_1^1(D) \cap C^1(D)$. We use the power series of the exponential function

$$\begin{aligned} &\int_D \exp A_1 \left(\exp A_2 \left(\frac{|u(x) - u_D|}{I(D)}\right)^{n/(n-1)}\right) dx \\ &\simeq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \int_D \left(\exp A_2 \left(\frac{|u(x) - u_D|}{I(D)}\right)^{n/(n-1)}\right)^i dx \\ &\simeq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{(A_2 i)^j}{j!} \int_D \left(\frac{|u(x) - u_D|}{I(D)}\right)^{nj/(n-1)} dx. \end{aligned}$$

We fix i_0 and j_0 later. By Lemma 3.2 with $q = nj/(n-1)$,

$$\begin{aligned} &\int_D \left(\frac{|u(x) - u_D|}{I(D)}\right)^{nj/(n-1)} dx \\ &\leq (c(n) c_0^{-1})^{nj/(n-1)} |D| (\log(e+j))^j \\ &\quad \times \left(\frac{|D|^{1/n} a_1}{(\log(e+j))^j I(D)} + \frac{I(\{y \in D : a_1 \leq |\nabla u(y)|\})}{I(D)}\right)^q. \end{aligned}$$

We choose a_1 so large that

$$\frac{I(\{y \in D : a_1 \leq |\nabla u(y)|\})}{I(D)} < \frac{1}{2^{(2n-1)/n} c(n) c_0^{-1}}.$$

Then we choose j_0 so large that

$$\frac{|D|^{1/n} a_1}{(\log(e+j))^j I(D)} < \frac{1}{2^{(2n-1)/n} c(n) c_0^{-1}}$$

for all $j \geq j_0 + 1$. Thus Lemma 2.8 implies, since we may choose i_0 so large that $A_2 i \geq 2e$ for all $i \geq i_0$,

$$\begin{aligned}
&\sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{(A_2 i)^j}{j!} \int_D \left(\frac{|u(x) - u_D|}{I(D)} \right)^{nj/(n-1)} dx \\
&\leq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{|D|}{j!} \left(\frac{A_2 i}{2} \right)^j (\log(e+j))^j \\
&\leq |D| \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \left(\frac{A_2 i}{2} \right)^{A_2 i} \leq |D| \sum_{i=0}^{\infty} \left(\frac{A_1 A_2}{2} \right)^i \frac{i^i}{i!} < \infty,
\end{aligned}$$

whenever $A_2 < 1$ and $A_1 A_2 < 2e^{-1}$. ∎

4. More irregular domains

Our goal is to give geometric criteria for domains where Sobolev inequalities of exponential type hold. We need the following definitions in our decomposition theorems.

Subsets $A_0, A_1, \ldots, A_k$ of R^n are said to form a chain, written as $\mathcal{C}(A_k) = (A_0, A_1, \ldots, A_k)$, if and only if $A_i \cap A_j \neq \emptyset$ whenever $|i - j| \leq 1$.

Let $\mathcal{W}$ be a family of domains $D \subset R^n$. We say that $\mathcal{W}$ is a decomposition of $G = \bigcup_{D \in \mathcal{W}} D$ if

$$\sum_{D \in \mathcal{W}} \chi_D(x) \leq c_1$$

for all $x \in R^n$ and there is a domain $D_0 \in \mathcal{W}$ such that for each $D \in \mathcal{W}$ there is a chain $\mathcal{C}(D) = (D_0, D_1, \ldots, D_k)$ of domains D_i in $\mathcal{W}$ with

$$\max\{|D_i|, |D_{i+1}|\} \leq c_2 |D_i \cap D_{i+1}|$$

for $i = 0, 1, \ldots, k-1$; here c_i, $i = 1, 2$, are constants.

An example of a decomposition of a domain G is its Whitney decomposition when we take dilated Whitney cubes. We consider this special case in Section 5.

Further, for a fixed set $A \in \mathcal{W}$ we write

$$A(\mathcal{W}) = \{D \in \mathcal{W} | A \in \mathcal{C}(D)\}.$$

We present a decomposition theorem which we later apply to more irregular domains than John domains.

THEOREM 4.1: *Let $\mathcal{W}$ be a family of c_0- John domains D in R^n. Let $\mathcal{W}$ be a decomposition of the domain $G = \bigcup_{D \in \mathcal{W}} D$ where G has finite n-measure. Suppose that*

$$\sum_{D \in \mathcal{A}(\mathcal{W})} \int_D \#\{A : A \in \mathcal{C}(D)\}^{q-1} \, dx \le c_3(G)^q |A|^{1/n(n-1/q)}$$

for $A \in \mathcal{W}$, where, $q \ge q_0(D)$ and c_1, c_2, and $c_3(G)$ are constants. Then there exist constants $c_i = c_i(c_1, c_2, c_3, |G|, n)$, $i = 4, 5$, such that

$$\tag{4.1} ⨍_G \exp\left(\frac{|u(x) - u_G|}{c_4 I(a, G)}\right)^\alpha dx \le c_5,$$

whenever $u \in W_1^1(G)$ satisfies $I(a, G) < \infty$, $\alpha = n/(n - 1 - an)$, and $a < 0$.

If $a = 0$, it is enough to assume that

$$\sum_{D \in \mathcal{A}(\mathcal{W})} \int_D \#\{A : A \in \mathcal{C}(D)\}^{q-1} \, dx \le c_6(G)$$

for all $A \in \mathcal{W}$.

Remarks:

1. A union of John domains in Theorem 4.1 need not be a John domain. We consider a large class of irregular non-John domains in Section 5; they satisfy the so-called quasi-hyperbolic boundary condition. J. Väisälä has considered conditions on John domains when the John property is preserved under a union [24].

2. A union of bounded c_o-John domains in Theorem 4.1 could be an unbounded non-John domain with finite n-measure. There is a planar tunnel example due to Väisälä:

$$G = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^i} B_i^j, \quad \text{where } B_i^j = B^2\left(i - 1 + \frac{j-1}{2^i}, \frac{1}{2^i}\right),$$

$j = 1, \ldots, 2^i$ and $i = 1, 2, \ldots$; here $B^2(x, r) = \{y \in R^2 : |x - y| < r\}$. Theorem 4.1 shows that the inequality of exponential type (4.1) holds for every $a \le 0$

in the tunnel domain G. Hence, in particular, the classical Trudinger inequality holds there, too. This shows that our decomposition theorem extends the class of domains where the Trudinger inequality is known to hold. Note that the domain G does not satisfy the quasi-hyperbolic boundary condition which we recall in Section 5.

Our proof makes use of the power series expansion of the exponential function and a chaining argument. As a starting point we need the following simple lemma.

LEMMA 4.1: *Let $D \subset R^n$ be a domain with finite n-measure. Let $F : [0,\infty) \to [0,\infty)$ be a convex, continuous, increasing function. Then for all u from an appropriate class and all $c \in R$,*

$$\int_D F(|u(x) - u_D|)\,dx \le \int_D F(2|u(x) - c|)\,dx.$$

Proof: The claim follows from the definition of convexity and Jensen's inequality, [14, p. 440]. ∎

Proof of Theorem 4.1: Let $\mathcal{W}$ be a given decomposition of the domain $G \subset R^n$. If $D_0 \in \mathcal{W}$ is a fixed set, then for each set $D \in \mathcal{W}$ there is a chain of sets joining D_0 and D, abbreviated by $\mathcal{C}(D) = (D_0, D_1, \dots, D)$. We denote

$$I(a,G) = \Big(\int_G |\nabla u(y)|^n \log(e + |\nabla u(y)|)^{an}\,dy \Big)^{1/n},$$

and recall $\alpha = n/(n-1-an)$. We may assume that $I(a,G) \le 1$. It is enough to estimate the sum

$$\sum_{j=j_0}^{\infty} \frac{1}{j!} \big(\frac{\tau}{I(a,G)}\big)^{\alpha j} \fint_G |u(x) - u_{D_0}|^{\alpha j}\,dx, \tag{4.2}$$

where $\tau > 0$ will be given later and j_0 is a suitable integer which depends on G. Thus $q \ge \alpha j_0 \ge 1$. The triangle inequality yields

$$\begin{aligned} \int_G |u(x) - u_{D_0}|^{\alpha j}\,dx &\le \sum_{D \in \mathcal{W}} \int_D |u(x) - u_{D_0}|^{\alpha j}\,dx \\ &\le 2^{\alpha j} \Big(\sum_{D \in \mathcal{W}} \int_D |u(x) - u_D|^{\alpha j}\,dx + \sum_{D \in \mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\,dx \Big). \end{aligned} \tag{4.3}$$

Let $\sigma = -an$ and $a < 0$. Recall that

$$I(a,D) = \Big(\int_D |\nabla u(x)|^n \log^{-\sigma}(e + |\nabla u(x)|) \Big)^{1/n}.$$

The John domain estimates (3.5) and (3.7) imply

$$\int_D |u(x)-u_D|^{\alpha j}\,dx \le c(n,a)^{\alpha j}(\alpha j)^j|D|(((|D|^{-1/(n^2-1)}+1)I(a,D))^{\alpha j}.$$

Hence,

$$\sum_{D\in\mathcal{W}}\int_D |u(x)-u_D|^{\alpha j}\,dx \le c(n,a,G)^{\alpha j}(\alpha j)^j|G|I(a,G)^{\alpha j}. \tag{4.4}$$

To estimate the sum

$$\sum_{D\in\mathcal{W}}\int_D |u_D-u_{D_0}|^{\alpha j}\,dx$$

we apply a chaining argument and obtain

$$\begin{aligned}\sum_{D\in\mathcal{W}}\int_D |u_D-u_{D_0}|^{\alpha j}\,dx &\le \sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}|u_{D_i}-u_{D_{i+1}}|\Big)^{\alpha j}dx\\ &\preceq \sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}\frac{1}{|D_i|}\int_{D_i}|u(y)-u_{D_i}|\,dy\Big)^{\alpha j}dx.\end{aligned}$$

A modification of Lemma 2.3 gives

$$\begin{aligned}&\Big(\int_{D_i}|u(x)-u_{D_i}|^{1/\epsilon}\,dx\Big)^{\epsilon}\epsilon^{1/\alpha}|D_i|^{-\epsilon-1/n+1/(n-\epsilon)}\\ &\le c(n)c_0^{-1}\sup_{0<\epsilon\le 1}\Big(\epsilon^{\sigma}\int_{D_i}|\nabla u(x)|^{n-\epsilon}\,dx\Big)^{1/(n-\epsilon)}.\end{aligned}$$

By using estimate (3.3) on the set $\{x : |\nabla u(x)|\ge e\}$ and estimate (3.6) on the set $\{x : |\nabla u(x)|\le e\}$ we obtain

$$\Big(\int_{D_i}|u(x)-u_{D_i}|^{1/\epsilon}\,dx\Big)^{\epsilon}\epsilon^{1/\alpha}|D_i|^{-\epsilon-1/n+1/(n-\epsilon)}\le c(n,a,c_0,G)I(a,D_i).$$

Hölder's inequality with $(\alpha j, \frac{\alpha j}{\alpha j-1})$ and the above estimates yield

$$\begin{aligned}&\frac{1}{|D_i|}\int_{D_i}|u(y)-u_{D_i}|\,dy \le |D_i|^{-1+1-1/\alpha j}\Big(\int_{D_i}|u(y)-u_{D_i}|^{\alpha j}\,dy\Big)^{1/\alpha j}\\ &\le c(n,a,c_0,G)\epsilon^{-1/\alpha}|D_i|^{1/n-1/(n-\epsilon)}I(a,D_i),\end{aligned}$$

where $\alpha j = q = \epsilon^{-1}$. Thus

$$\begin{aligned}&\sum_{D\in\mathcal{W}}\int_D |u_D-u_{D_0}|^{\alpha j}\,dx\\ &\le c(G)^{\alpha j}(\alpha j)^j\sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}|D_i|^{1/n-1/(n-\epsilon)}I(a,D_i)\Big)^{\alpha j}dx.\end{aligned}$$

We show that

$$\sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}|D_i|^{1/n-1/(n-1/q)}I(a,D_i)\Big)^{\alpha j}dx\le cI(a,G)^q \tag{4.6}$$

for some $c=c(n,a,c_0,G)^q$. When we change the order of summation on the left hand side of the above inequality we obtain

$$\begin{aligned}&\sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}|D_i|^{1/n-1/(n-1/q)}I(a,D_i)\Big)^{q}dx\\ &\le\sum_{D\in\mathcal{W}}\int_D\#\{B|B\in\mathcal{C}(D)\}^{q-1}\,dx\sum_{A\in\mathcal{C}(D)}\Big(|A|^{1/n-1/(n-1/q)}\Big)^qI(a,A)^q\\ &=\sum_{A\in\mathcal{W}}\sum_{D\in\mathcal{A}(\mathcal{W})}\int_D\#\{B|B\in\mathcal{C}(D)\}^{q-1}\,dx|A|^{-1/n(n-1/q)}I(a,A)^q.\end{aligned}$$

Hence, to obtain

$$\sum_{D\in\mathcal{W}}\int_Q\Big(\sum_{i=0}^{k}|D_i|^{1/n-1/(n-1/q)}I(a,D_i)\Big)^{q}dx\le c(G)^qI(a,G)^q,$$

it is enough to have

$$\sum_{D\in\mathcal{A}(\mathcal{W})}\int_D\#\{B|B\in\mathcal{C}(D)\}^{q-1}\,dx\le c|A|^{1/n(n-1/q)}$$

for some $c<\infty$, which was the assumption.

Combination of estimates (4.2)–(4.6) gives us

$$\int_G|u(x)-u_{D_0}|^{\alpha j}\,dx=\sum_{D\in\mathcal{W}}\int_D|u(x)-u_{D_0}|^{\alpha j}\,dx\le c(G)^{\alpha j}(\alpha j)^jI(a,G)^{\alpha j}.$$

The end of the proof is similar to the end of the proof for Theorem 3.2. The proof is complete in the case $a<0$.

If $a=0$, we apply Hölder's inequality with $(\frac{n}{n-\epsilon},\frac{n}{\epsilon})$ to the John domain estimate of Lemma 2.3. Thus, we have

$$\begin{aligned}
&\sum_{D\in\mathcal{W}}\int_D |u_D-u_{D_0}|^{\alpha j}\,dx\\
&\le c(G)^{\alpha j}(\alpha j)^j\sum_{D\in\mathcal{W}}\int_D\Big(\sum_{i=0}^{k}|D_i|^{1/n-1/(n-\epsilon)+\epsilon/n(n-\epsilon)}\Big(\int_{D_i}|\nabla u(x)|^n\,dx\Big)^{1/n}\Big)^{\alpha j}\\
&\le c(G)^{\alpha j}(\alpha j)^j\sum_{A\in\mathcal{W}}\sum_{D\in\mathcal{A}(\mathcal{W})}\int_D \#\{B|B\in\mathcal{C}(D)\}^{q-1}\,dx\Big(\int_A|\nabla u(x)|^n\,dx\Big)^{q/n}\\
&\le c(G)^q\Big(\int_G|\nabla u(x)|^n\,dx\Big)^{q/n}
\end{aligned}$$

by the assumption. Hence we have estimated the latter part, when $a = 0$. Since the first part follows immediately by Hölder's inequality, the case $a = 0$ follows. ∎

We show in Section 5 that there is a large class of domains which satisfy the requirements of the decomposition theorem.

THEOREM 4.2: *Let $\mathcal{W}$ be a family of c_0-John domains D in R^n. Let $\mathcal{W}$ be a decomposition of the domain $G = \bigcup_{D\in\mathcal{W}} D$ where G has finite n-measure. Suppose that*

$$\sum_{D\in\mathcal{W}}\int_D \#\{A\in\mathcal{W} : A\in\mathcal{C}(D)\}^{\alpha j}\,dx \le c_3(G,a,n)^j$$

where $\alpha = n/(n-1-an)$, and $a\in[0,1-1/n)$, for all $j\ge j_0$.

Then there are constants $c_4>0$ and c_5 such that

$$⨍_G \exp\Big(\frac{|u(x)-u_G|}{c_4 I(a,G)}\Big)^{\alpha}\,dx \le c_5 < \infty,$$

whenever $u\in W_1^1(G)$ satisfies $I(a,G)<\infty$.

Proof of Theorem 4.2: Let $\mathcal{W}$ be a given decomposition of the domain $G\subset R^n$. If $D_0\in\mathcal{W}$ is a fixed set, then for each set $D\in\mathcal{W}$ there is a chain of sets joining D_0 and D, abbreviated by $\mathcal{C}(D) = (D_0, D_1, \dots, D)$. It is enough to estimate the sum

$$\sum_{j=j_0}^{\infty}\frac{1}{j!}\Big(\frac{\tau}{I(a,G)}\Big)^{\alpha j}⨍_G |u(x)-u_{D_0}|^{\alpha j}\,dx,$$

where $\tau > 0$ will be given later and j_0 is a suitable integer which depends on G. Thus $q \geq \alpha j_0 \geq 1$. The triangle inequality yields

$$\int_G |u(x) - u_{D_0}|^{\alpha j}\,dx \leq \sum_{D\in\mathcal{W}} \int_D |u(x) - u_{D_0}|^{\alpha j}\,dx$$
$$\leq 2^{\alpha j}\Big(\sum_{D\in\mathcal{W}} \int_D |u(x) - u_D|^{\alpha j}\,dx + \sum_{D\in\mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\,dx\Big).$$

The John domain estimate in Lemma 3.1 for $2 < a_1$ implies

$$\Big(\int_D |u(x) - u_D|^{\alpha j}\,dx\Big)^{1/\alpha j}$$
$$\leq (c_0^{-1}c(n))|D|^{1/\alpha j}\Big(|D|^{1/n}a_1$$
$$+ 2\frac{(\alpha j)^{1/\alpha}(n-1)^{(n-1)/n}}{(n-1-na)^{(n-1)/n}} I(a, \{y \in D : a_1 \leq |\nabla u(y)|\})\Big)$$
$$\leq c_0^{-1}c(a,n)|D|^{1/\alpha j} j^{1/\alpha}\Big(\frac{|G|^{1/n}a_1}{j^{1/\alpha}} + I(a, G)\Big).$$

We choose j_0 so large that

$$\frac{|G|^{1/n}a_1}{j^{1/\alpha}} \leq I(a, G)$$

with $j \geq j_0$. Hence,

$$\sum_{D\in\mathcal{W}} \int_D |u(x) - u_D|^{\alpha j}\,dx \leq (c_0^{-1}c(n,a))^{\alpha j} j^j |G| I(a, G)^{\alpha j}.$$

To estimate the sum

$$\sum_{D\in\mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\,dx$$

we apply a chaining argument plus Hölder's inequality with $(\alpha j, \frac{\alpha j}{\alpha j - 1})$ and obtain

$$\sum_{D\in\mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\,dx$$
$$\leq \sum_{D\in\mathcal{W}} \int_D \Big(\sum_{i=0}^{k} |u_{D_i} - u_{D_{i+1}}|\Big)^{\alpha j}\,dx$$
$$\leq 2 \sum_{D\in\mathcal{W}} \int_D \Big(\sum_{i=0}^{k} \frac{1}{|D_i|} \int_{D_i} |u(y) - u_{D_i}|\,dy\Big)^{\alpha j}\,dx$$
$$\leq 2 \sum_{D\in\mathcal{W}} \int_D \Big(\sum_{i=0}^{k} \frac{|D_i|^{1-1/\alpha j}}{|D_i|} \Big(\int_{D_i} |u(y) - u_{D_i}|^{\alpha j}\,dy\Big)^{1/\alpha j}\Big)^{\alpha j}\,dx.$$

Lemma 3.1 for $2 < a_1$ as above gives

$$\begin{aligned}&\sum_{i=0}^{k} \frac{1}{|D_i|^{1/\alpha j}} \Big(\int_{D_i} |u(y) - u_{D_i}|^{\alpha j}\, dy \Big)^{1/\alpha j} \\ &\le \sum_{i=0}^{k} c_0^{-1} c(n) \Big(a_1 |D|^{1/n} + \Big(\Big(\frac{n-1}{n-1-an} \Big)^{(n-1)/n} (\alpha)^{1/\alpha} \Big) j^{1/\alpha} I(a,G) \\ &\le c_0^{-1} c(n) \#\{A \in \mathcal{W} : A \in \mathcal{C}(D_k)\} c(n,a) j^{1/\alpha} I(a,G).\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{D \in \mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\, dx \\ &\le (c_0^{-1} c(n) c(n,a))^{\alpha j} \Big(\sum_{D \in \mathcal{W}} \int_D \#\{A \in \mathcal{W} : A \in \mathcal{C}(D_k)\}^{\alpha j}\, dx \Big) j^j I(a,G)^{\alpha j}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{D \in \mathcal{W}} \int_D |u_D - u_{D_0}|^{\alpha j}\, dx \\ &\le (c_0^{-1} c(n,a))^{\alpha j} \Big(|G| + \sum_{D \in \mathcal{W}} \int_D \#\{A \in \mathcal{W} : A \in \mathcal{C}(D_k)\}^{\alpha j}\, dx \Big) j^j I(a,G)^{\alpha j},\end{aligned}$$

and further,

$$\begin{aligned}&\sum_{j=j_0}^{\infty} \frac{1}{j!} \Big(\frac{\tau}{I(a,G)} \Big)^{\alpha j} \fint_G |u(x) - u_{D_0}|^{\alpha j}\, dx \\ &\le \sum_{j=j_0}^{\infty} \frac{1}{j!} (\tau c_0^{-1} c(n,a))^{\alpha j} j^j \Big(1 + \frac{\sum_{D \in \mathcal{W}} \int_D \#\{A \in \mathcal{W} : A \in \mathcal{C}(D)\}^{\alpha j}\, dx}{|G|} \Big) \\ &\le \sum_{j=j_0}^{\infty} \frac{(\tau c_0^{-1} c(a,n) c(D))^{\alpha j}}{j!} j^j.\end{aligned}$$

As soon as we choose τ small enough the claim follows. ∎

5. Conditions with quasi-hyperbolic metric

We consider a general example of the decomposition we defined in Section 4, namely a Whitney decomposition $\mathcal{W}$ of a given domain D. This decomposition holds for all domains. By using a Whitney decomposition we obtain a geometric criterion for the validity of Trudinger's inequality in an arbitrary domain with finite n-measure.

If Q_0 is a fixed Whitney cube, then for each Whitney cube Q, there is a chain of dilated Whitney cubes joining $\frac{9}{8}Q_0$ and $\frac{9}{8}Q$, $\mathcal{C}(\frac{9}{8}Q) = (\frac{9}{8}Q_0, \frac{9}{8}Q_1, \ldots, \frac{9}{8}Q)$. We drop off $\frac{9}{8}$ from the front of Q just to keep the calculations neater. From now on, we denote

$$\mathcal{W} = \{\frac{9}{8}Q | Q \text{ is a Whitney cube}\}.$$

The chain $\mathcal{C}(Q)$ can be chosen so that the number of cubes in this chain $\mathcal{C}(Q)$, $Q \in \mathcal{W}$, of dilated Whitney cubes can be estimated by means of the quasi-hyperbolic distance between points $x_1, x_2 \in D$, defined by

$$k_D(x_1, x_2) = \inf_\gamma \int_\gamma \frac{dz}{d(x, \partial D)},$$

where the infimum is taken over all rectifiable curves $\gamma\colon [0,l] \to D$, parametrized by arc length, joining x_1 to x_2 in D. The number of cubes in $\mathcal{C}(Q)$ is comparable to $k_D(Q_0, Q)$.

LEMMA 5.1 ([13, Proposition 6.1]): *Let $\mathcal{W}$ be a Whitney decomposition of a domain $D \subset R^n$. If $Q_0 \in \mathcal{W}$ and $x_0 \in Q_0$ are fixed, then for each $Q \in \mathcal{W}$ there is a chain $\mathcal{C}(Q)$ of dilated Whitney cubes joining Q_0 and Q such that for all $x \in Q$*

$$\#\{B \in \mathcal{W} | B \in \mathcal{C}(Q)\} \leq c(n) k_D(x_0, x) + 1 \leq 5c(n)(\#\{B \in \mathcal{W} | B \in \mathcal{C}(Q)\} + 1).$$

Further, we write

$$\mathcal{A}(\mathcal{W}) = \{Q \in \mathcal{W} \colon A \in \mathcal{C}(Q)\}$$

for each $A \in \mathcal{W}$ on a domain D.

We give a general condition for a domain D which enables us to show that a Trudinger-type inequality holds in D, Theorem 5.1. We show in Theorem 5.2 that there are domains which satisfy the condition given in Theorem 5.1.

THEOREM 5.1: *Let $D \subset R^n$ be a domain such that*

$$\sum_{Q \in \mathcal{A}(\mathcal{W})} \int_Q k_D(x_0, Q)^{q-1}\, dx \leq c|A|^{1/n(n-1/q)}$$

for $A \in \mathcal{W}$. Here, $q \geq 1$ and $c \leq c(D)^q$ is a constant. Then there exist constants $c_i = c_i(a, c_0, |D|, n)$, $i = 1, 2$, such that

$$⨍_D \exp\left(\frac{|u(x) - u_D|}{c_1 I(a, D)}\right)^\alpha dx \leq c_2,$$

whenever $u \in W_1^1(D)$ satisfies $I(a, D) < \infty$, $\alpha = n/(n-1-an)$, and $a < 0$.

If $a = 0$, it is enough to assume that

$$\sum_{Q\in A(\mathcal{W})} \int_Q k_D(x_0, Q)^{q-1}\, dx \le c$$

for all $A \in \mathcal{W}$. Here, $c \le c(D)^q$ is a constant.

Proof: Just use Theorem 4.1 and Lemma 5.1. ∎

Remark: Domains which satisfy the conditions in Theorem 5.1 are (p,p)-Poincaré domains with some p by [13, Theorem 6.6].

QUASI-HYPERBOLIC BOUNDARY CONDITION DOMAINS.

We show that a quasi-hyperbolic boundary condition domain D with a constant b satisfies

$$\sum_{Q\in A(\mathcal{W})} \int_Q k_D(x_0, Q)^{q-1}\, dx \le c(D)^q |A|^{1/2b-\delta},$$

where $\delta > 0$ is arbitrarily small.

A domain is a quasi-hyperbolic boundary condition domain, if there exist $x_0 \in D$ and $b \ge 1$ such that

$$k_D(x_0, x) \le b \log\left(1 + \frac{|x - x_0|}{\min\{d(x, \partial D), d(x_0, \partial D)\}}\right) \tag{5.1}$$

for all $x \in D$.

John domains form a proper subclass of quasi-hyperbolic boundary condition domains. An example of a non-John domain which satisfies (5.1) is a domain which is formed from a square to which an infinite number of small triangles are attached. An example is presented in [10, Example 2.26].

A quasi-hyperbolic boundary condition domain satisfies a Whitney cube #-condition, [20, Corollary 2] and [17, Lemma 3.9]. This means that there exists $k_0 \ge 1$ such that for all $k \ge k_0$

$$\#\{Q \in \mathcal{W} \mid |Q| \simeq 2^{-nk}|D|\} \le c(D)2^{\lambda k},$$

where $\lambda < n$.

We need the following lemma.

LEMMA 5.2: *Let $D \subset R^n$ be a quasi-hyperbolic boundary condition domain with a constant b. Let A be a cube in a Whitney decomposition of D. Suppose that $1 \le q$. Then there is a constant $c = c(D)$ such that*

$$\sum_{Q\in A(\mathcal{W})} \int_Q k_D(x_0, x)^{q-1}\, dx \le c^q |A|^{1/2b-\delta}$$

where $0 < \delta$ is arbitrarily small.

Proof: Fix A in a Whitney decomposition of the domain D. By [13, Lemma 7.25]

$$|A|^{1/2b} \geq |Q|$$

for all $Q \in A(\mathcal{W})$. We write

$$B_j = \{Q \in A(\mathcal{W}) \, |2^{-(j+1)n} \leq |Q|/|A|^{1/2b} \leq 2^{-jn}\}$$

for $j = 1, \ldots$. Since D satisfies a Whitney cube #-condition, we obtain

$$\begin{aligned}\sum_{Q \in A(\mathcal{W})} \int_Q k_D(x_0, x)^{q-1}\, dx &\leq \sum_{j=1}^{\infty} \sum_{B_j} \int_Q k_D(x, x_0)^{q-1}\, dx \\ &\leq c(b, D)^q |A|^{1/2b} \sum_{j=1}^{\infty} j^q 2^{(\lambda - n)j} \\ &\leq c(b, D)^q |A|^{1/2b - \delta},\end{aligned}$$

where $\lambda < n$ and $\delta > 0$ is arbitrarily small. ∎

Remark: A domain D satisfying a quasi-hyperbolic condition is a (p_0, p_0)-Poincaré domain where $p_0 < n$ is the Minkowski dimension of the boundary of D, [6, Remark 3.8]. Hence, by Remark 2.4(3) it is enough to consider function $u \in C^1(D) \cap W_1^1(D)$ and as soon as $I(a, D) < \infty$, and by Remark 2.4(3) the integral average of a C^1-function over the domain is finite as soon as $I(a, D) < \infty$.

THEOREM 5.2: *Let $D \subset R^n$ be a quasi-hyberbolic boundary condition domain with a constant b. Then there exist constants $c_i = c_i(a, c_0, D, n)$, $i = 1, 2$, such that*

$$\int_D \exp\biggl(\frac{|u(x) - u_D|}{c_1 I(a, D)}\biggr)^{\alpha} dx \leq c_2,$$

whenever $I(a, D) < \infty$, $\alpha = n/(n - 1 - an)$ and $a \leq 0$.

Proof: Lemma 5.2 implies that if $q > 1$,

$$\sum_{Q \in \mathcal{A}(\mathcal{W})} \int_Q k_D(x_0, Q)^{q-1}\, dx \leq c_1^q |A|^{1/2b - \delta},$$

where $\delta > 0$ is arbitrarily small. The constant $c_1 = c_1(D)$ does not depend on q. According to Remark 2.4(1) it is enough to have j_0 such that $q = \alpha j_0 \geq 1$.

Hence for a quasi-hyperbolic boundary condition domain D we obtain

$$\sum_{A\in\mathcal{W}}\sum_{Q\in\mathcal{A}(\mathcal{W})}\int_Q k_D(x_0,Q)^{q-1}\,dx|A|^{-1/n(n-1/q)}(I(a,A))^q$$
$$\leq c_1^q\sum_{A\in\mathcal{W}}|A|^{1/2b-\delta}|A|^{-1/n(n-1/q)}(I(a,A))^q.$$

We may assume $|A|\leq 1$. Thus,

$$\sum_{A\in\mathcal{W}}\sum_{Q\in\mathcal{A}(\mathcal{W})}\int_Q k_D(x_0,Q)^{q-1}\,dx|A|^{-1/n(n-1/q)}(I(a,A))^q\leq c_1^q(I(a,D))^q,$$

whenever

$$q\geq\frac{n(1/2b-\delta)}{n^2(1/2b-\delta)-1}$$

and $\delta>0$ is arbitrarily small. This means a lower bound for j which is allowed by Remark 2.4(1). This yields the claim, when $a<0$.

If $a=0$, the claim follows more easily since it is enough to assume

$$\sum_{Q\in\mathcal{A}(\mathcal{W})}\int_Q k_D(x,x_0)^{q-1}\,dx\leq c(D)^q.$$

Remark: As we showed in Remarks 2 in Section 4, there are domains other than those with a quasi-hyperbolic boundary condition where Sobolev inequalities of exponential type hold and, in particular, the Trudinger inequality holds.

THEOREM 5.3: *Let $D\subset R^n$ be a bounded domain. Suppose that*

$$\int_D k_D(x_0,x)^q\,dx\leq c(D)^q<\infty$$

for all $q>n$. Then for any constant $A>0$

$$⨍_D \exp A\left(\frac{|u(x)-u_D|}{I(a,D)}\right)^\alpha dx<\infty,$$

where $\alpha=n/(n-1-an)$ with $a\in[0,1-1/n)$, whenever $u\in W_1^1(D)$ satisfies $I(a,D)<\infty$.

Proof: The proof for Theorem 5.2 yields the claim for some constant A, since for each $D\in\mathcal{W}$

$$\#\{A\in\mathcal{W}:A\in\mathcal{C}(D)\}\simeq c(n)k_D(x_0,x)$$

where $x_0\in D_0$ and $x\in D$. The fact that the claim is true for any A follows as in the proof of Theorem 3.3. ∎

THEOREM 5.4: *Let $D \subset R^n$ be a bounded domain. Suppose that*

$$\int_D k_D(x_0, x)^q \, dx \le c(D)^q < \infty$$

for all $q > n$ and let $u \in W_1^1(D)$ satisfy $I(D) := I(1 - 1/n, D) < \infty$. Then given positive constants A_1 and $A_2 < 1$ such that $A_1 A_2 < 2e^{-1}$ there is $A_3 > 0$, which does not depend on the gradient of u, such that

$$\int_D \exp A_1 \left(\exp A_2 \left(\frac{|u(x) - u_D|}{I(D)} \right)^{n/(n-1)} \right) dx \le A_3 < \infty.$$

Remark: It is well known, [13, Theorem 7.7] and [20, Corollary 11], that for domains D satisfying a quasi-hyperbolic boundary condition

$$\int_D k_D(x_0, x)^q \, dx \le c(D)^q < \infty$$

for all $q > n$.

Proof of Theorem 5.4: The power series of the exponential function yields

$$\begin{aligned}
&\int_D \exp A_1 (\exp A_2 \left(\frac{|u(x) - u_{Q_0}|}{I(D)} \right)^{n/(n-1)}) \, dx \\
&\simeq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \int_D \left(\exp A_2 \left(\frac{|u(x) - u_{Q_0}|}{I(D)} \right)^{n/(n-1)} \right)^i dx \\
&\simeq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{(A_2 i)^j}{j!} \int_D \left(\frac{|u(x) - u_{Q_0}|}{I(D)} \right)^{nj/(n-1)} dx \\
&\le \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{(A_2 i)^j}{j!} 2^{nj/(n-1)} \Bigg(\sum_{Q \in \mathcal{W}} \int_Q \left(\frac{|u(x) - u_Q|}{I(D)} \right)^{nj/(n-1)} dx \\
&+ \sum_{Q \in \mathcal{W}} \int_Q \left(\frac{|u_Q - u_{Q_0}|}{I(D)} \right)^{nj/(n-1)} dx \Bigg).
\end{aligned}$$

Lemma 3.1 yields an estimate of the sum

$$\sum_{Q \in \mathcal{W}} \int_Q \left(\frac{|u(x) - u_Q|}{I(D)} \right)^{nj/(n-1)} dx.$$

We choose $q = nj/(n-1)$ in Lemma 3.2. Thus

$$\sum_{Q \in \mathcal{W}} \int_Q \left(\frac{|u(x) - u_Q|}{I(D)} \right)^{nj/(n-1)} dx$$

$$\leq \sum_{Q\in\mathcal{W}} c(n)^{nj/(n-1)}|Q|(\log(e+j))^j$$
$$\times \left(\frac{|D|^{1/n}a_1}{(\log(e+j))^j I(D)} + \frac{I(\{y\in D : a_1 \leq |\nabla u(y)|\})}{I(D)}\right)^q.$$

We choose a_1 so that

$$\frac{I(\{y\in D : a_1 \leq |\nabla u(y)|\})}{I(D)} \leq \frac{1}{2^{(3n-1)/n}c(n)}.$$

We choose j_0 large enough so that

$$\frac{|D|^{1/n}a_1}{(\log(e+j))^j I(D)} \leq \frac{1}{2^{(3n-1)/n}c(n)}.$$

Thus,

$$\sum_{Q\in\mathcal{W}} \int_Q \left(\frac{|u(x)-u_Q|}{I(D)}\right)^{nj/(n-1)} dx \leq \sum_{Q\in\mathcal{W}} \frac{|Q|}{2^{(1+n/(n-1))j}}(\log(e+j))^j.$$

Hence by Lemma 2.8, for i_0 so large that $A_2 i > 2e$ with all $i \geq i_0$,

$$\sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{(A_2 2^{n/(n-1)}i)^j}{j!} \int_D \left(\frac{|u(x)-u_D|}{I(D)}\right)^{nj/(n-1)} dx$$
$$\leq \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!} \sum_{j=j_0}^{\infty} \frac{|D|}{j!}\left(\frac{A_2 i}{2}\right)^j (\log(e+j))^j$$
$$\leq |D| \sum_{i=i_0}^{\infty} \frac{A_1^i}{i!}\left(\frac{A_2 i}{2}\right)^i < \infty,$$

whenever $A_2 < 1$ and $A_1 A_2 < 2e^{-1}$.

The estimation of the second sum

$$\sum_{Q\in\mathcal{W}} \int_D \left(\frac{|u_Q - u_{Q_0}|}{I(D)}\right)^{nj/(n-1)} dx$$

is more complicated. By using a chaining argument and Hölder's inequality with

$(q, \frac{q}{q-1})$ we obtain, with $Q =: Q_k$,

$$\begin{aligned}
&\sum_{Q\in\mathcal{W}}\int_D |u_Q - u_{Q_0}|^{nj/(n-1)}\,dx \\
&\le \sum_{Q\in\mathcal{W}}\int_Q\Big(\sum_{i=i}^{k-1}|u_{Q_i} - u_{Q_{i+1}}|\Big)^{nj/(n-1)}\,dx \\
&\le 2\sum_{Q\in\mathcal{W}}\int_Q\Big(\sum_{i=0}^{k}\frac{1}{|Q_i|}\int_{Q_i}|u(y) - u_{Q_i}|\,dy\Big)^{nj/(n-1)}\,dx \\
&\le 2\sum_{Q\in\mathcal{W}}\int_Q\Big(\sum_{i=0}^{k}\frac{1}{|Q_i|^{1/q}}\Big(\int_{Q_i}|u(y) - u_{Q_i}|^q\,dy\Big)^{1/q}\Big)^{nj/(n-1)}\,dx.
\end{aligned}$$

As in the first case with $q = nj/(n-1)$,

$$\begin{aligned}
&\sum_{s=0}^{k}\frac{1}{|Q_s|^{1/q}}\Big(\int_{Q_s}|u(x) - u_{Q_s}|^q\,dx\Big)^{1/q} \\
&\le \sum_{s=0}^{k} 2^{-2+1/n}|Q_s|^{1/q-1/q}(\log(e+j))^{(n-1)/n} \\
&\quad\times\Big(\frac{|D|^{1/n}a_1}{(\log(e+j))^j I(D)} + \frac{I(\{y\in D : a_1 \le |\nabla u(y)|\})}{I(D)}\Big).
\end{aligned}$$

We choose a_1 so that

$$\frac{I(\{y\in D : a_1 \le |\nabla u(y)|\})}{I(D)} \le \frac{1}{2^2(c_1(n)c(D) + |D|)},$$

where $c_1(n)$ is the constant from Lemma 5.1. and $c(D)$ is the constant from the assumption. We choose j_0 large enough so that

$$\frac{|D|^{1/n}a_1}{(\log(e+j))^j I(D)} \le \frac{1}{2^2(c_1(n)c(D) + |D|)}.$$

Thus,

$$\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\sum_{j=j_0}^{\infty}\frac{(A_2 2^{n/(n-1)}i)^j}{j!}\sum_{Q\in\mathcal{W}}\int_Q\left(\frac{|u_Q-u_{Q_0}|}{I(D)}\right)^{nj/(n-1)}dx$$

$$\leq 2\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\sum_{j=j_0}^{\infty}\frac{(A_2 2^{n/(n-1)}i)^j}{j!}$$

$$\times\sum_{Q\in\mathcal{W}}\int_Q\left(\sum_{A\in\mathcal{C}(Q)}(2^{(3n-1)/n}(c_1(n)c(D)+|D|))^{-1}(\log(e+j))^{(n-1)/n}\right)^q dx$$

$$\leq 2\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\sum_{j=j_0}^{\infty}\frac{(A_2 i)^j}{(2(2c_1(n)c(D)+2|D|)^{n/(n-1)})^j j!}$$

$$\times\sum_{Q\in\mathcal{W}}\int_Q(c_1(n)k_D(x_0,x)+1)^q\,dx(\log(e+j))^j$$

$$\leq 4\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\sum_{j=j_0}^{\infty}\frac{(A_2 i)^j}{2^j j!}(\log(e+j))^j.$$

By Lemma 2.8, when we choose i_0 so large that $A_2 i > 2e$, we obtain

$$\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\sum_{j=j_0}^{\infty}\frac{1}{j!}(\frac{A_2 i}{2})^j(\log(e+j))^j\leq\sum_{i=i_0}^{\infty}\frac{A_1^i}{i!}\left(\frac{A_2 i}{2}\right)^i<\infty,$$

whenever $A_2 < 1$ and $A_1 A_2 < 2e^{-1}$. The claim follows. ∎

References

[1] D. R. Adams, *A sharp inequality of J. Moser for higher order derivatives*, Annals of Mathematics **128** (1988), 385–398.

[2] R. A. Adams, *Sobolev Spaces*, Academic Press, Orlando, 1975.

[3] B. Bojarski, *Remarks on Sobolev imbedding inequalities*, Proceedings of the Conference on Complex Analysis, Joensuu, Lecture Notes in Mathematics **1351**, Springer-Verlag, Berlin, 1987, pp. 52–68.

[4] S. Buckley and J. O'Shea, *Weighted Trudinger-type inequalities*, Indiana University Mathematics Journal **48** (1999), 85–114.

[5] A. Cianchi, *Some results in the theory of Orlicz spaces and applications to variational problems*, in *Proceedings of the Spring School 'Nonlinear Analysis, Function Spaces and Applications', Prague, May 31–June 6, 1999*, Olympia Press, Prague, 1999, pp. 50–92.

[6] D. E. Edmunds and R. Hurri-Syrjänen, *Weighted Poincaré inequalities and Minkowski content,* Proceedings of the Royal Society of Edinburgh. Section A. Mathematics **125** (1995), 817–825.

[7] D. E. Edmunds, P. Gurka and B. Opic, *On embeddings of logarithmic Bessel potential spaces,* Journal of Functional Analysis **146** (1997), 116–150.

[8] W. D. Evans and D. J. Harris, *Sobolev embeddings for generalized ridged domains,* Proceedings of the London Mathematical Society **54** (1987), 141–175.

[9] N. Fusco, P. L. Lions and C. Sbordone, *Sobolev imbedding theorems in borderline cases,* Proceedings of the American Mathematical Society **124** (1996), 561–565.

[10] F. W. Gehring and O. Martio, *Lipschitz classes and quasiconformal mappings,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica **10** (1985), 203–219.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag Berlin–Heidelberg–New York, 1977.

[12] P. Hajlasz and P. Koskela, *Isoperimetric inequalities and imbedding theorems in irregular domains,* Journal of the London Mathematical Society (2) **58** (1998), 425–450.

[13] R. Hurri, *Poincaré domains in R^n,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica Dissertationes **71** (1988), 1–41.

[14] R. Hurri-Syrjänen, *An inequality of Bhattacharya and Leonetti,* Canadian Mathematical Bulletin **39** (1996), 438–447.

[15] O. Martio, *John domains, bilipschitz balls and Poincaré inequality,* Revue Roumaine de Mathématiques Pures et Appliquées **33** (1988), 107–112.

[16] O. Martio and J. Sarvas, *Injectivity theorems in plane and space,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica **4** (1978–79), 383–401.

[17] O. Martio and M. Vuorinen, *Whitney cubes, p-capacity, and Minkowski content,* Expositiones Mathematicae **5** (1987), 17–40.

[18] Y. Mizuta and T. Shimomura, *Exponential integrability for Riesz potentials of functions in Orlicz classes,* Hiroshima Mathematical Journal **28** (1998), 355–371.

[19] J. Moser, *A sharp form of an inequality by N. Trudinger,* Indiana University Mathematics Journal **20** (1971), 1077–1092.

[20] W. Smith and D. Stegenga, *Exponential integrability of the quasi-hyperbolic metric on Hölder domains,* Annales Academiae Scientiarum Fennicae. Series A I. Mathematica **16** (1991), 345–360.

[21] W. Smith and D. Stegenga, *Sobolev imbeddings and integrability of harmonic functions on Hölder domains,* in *Potential Theory (Nagoya, 1990),* de Gruyter, Berlin, 1992, pp. 303–313.

[22] R. S. Strichartz, *A note on Trudinger's extension of Sobolev inequality*, Indiana University Mathematics Journal **21** (1972), 841–842.

[23] N. S. Trudinger, *On imbeddings into Orlicz spaces and some applications*, Journal of Mathematics and Mechanics **17** (1967), 473–483.

[24] J. Väisälä, *Unions of John domains*, Proceedings of the American Mathematical Society **128** (2000), 1135–1140.